

A POSSIBLE GENERALIZATION OF THE HARMONIC OSCILLATOR POTENTIAL

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G. Lévai

*Institute of Nuclear Research of the Hungarian Academy of Sciences
Debrecen, PO Box 51, Hungary 4001*

Abstract

A four-parameter potential is analyzed, which contains the three-dimensional harmonic oscillator as a special case. This potential is exactly solvable and retains several characteristics of the harmonic oscillator, and also of the Coulomb problem. The possibility of similar generalizations of other potentials is also pointed out.

1 Introduction

Searching for exact solutions of the Schrödinger equation has been an interesting challenge since the early period of quantum mechanics. This classic area has gained new momentum from the recent introduction of supersymmetric quantum mechanics (SUSYQM) [1], which relates pairs of essentially isospectral potentials to each other by means of (super)algebraic manipulations. (See, for example [2] for a recent review on SUSYQM, and [3] and references for its relation to other methods of analyzing isospectral potentials.) This new approach helped to view old problems from a new angle, and allowed unified, systematic treatment of previously unrelated results. New solutions of the Schrödinger equation have been described and classified, together with already known ones. The most well-known potentials have been shown to have the property of shape invariance [4], a concept introduced in SUSYQM. Much less is known, however, about the more general Natanzon potentials [5], which are, in principle, solvable, nevertheless their practical use is hindered by their complicated mathematical structure. The techniques inspired by SUSYQM allow a straightforward generalization of the simplest shape-invariant potentials, while avoiding most of the mathematical complications characterizing the general Natanzon potentials.

Here I discuss a potential which can be considered the simultaneous generalization of the three-dimensional harmonic oscillator and Coulomb potentials: these two shape-invariant potentials can be obtained from it by tuning one of its four parameters. Its Coulomb limit has already been described [6], and here I discuss its connection with the harmonic oscillator. In contrast with other anharmonic oscillators, this potential converges to a finite value in the $r \rightarrow \infty$ limit. It also inherited several characteristics from its two "parent potentials", which may enable its applications to physical problems, where deviations from these two fundamental potentials are relevant.

In Section 2. I give a brief account of a simple procedure which can be used to derive exactly solvable potentials. Section 3. contains the main results of this contribution, while in Section 4. a summary is given and directions towards further investigations are pointed out.

2 Transformations of the Schrödinger Equation

Here I describe an old method of solving the Schrödinger equation to demonstrate how a wide range of solvable potentials can be derived in a relatively straightforward way. Originally this procedure was used [7] to derive only some well-known potentials, but it can be shown that the general Natanzon potentials can also be derived from it. This procedure was also connected to the formalism of SUSYQM [8].

The solutions of the one-dimensional Schrödinger equation (with $\hbar = 2m = 1$)

$$\frac{d^2\Psi}{dx^2} + (E - V(x))\Psi(x) = 0 \quad (1)$$

are generally written as

$$\Psi(x) = f(x)F(g(x)), \quad (2)$$

where $F(g)$ is a special function which satisfies a second-order differential equation

$$\frac{d^2F}{dg^2} + Q(g)\frac{dF}{dg} + R(g)F(g) = 0. \quad (3)$$

Here $Q(g)$ and $R(g)$ are well-known for any specified special function $F(g)$, while $f(x)$ and $g(x)$ are some functions to be determined. Substituting (2) in (1) and comparing the results with (3) we arrive at the following expression [8] after some straightforward algebra:

$$E - V(x) = \frac{g'''(x)}{2g'(x)} - \frac{3}{4} \left(\frac{g''(x)}{g'(x)} \right)^2 + (g'(x))^2 \left(R(g(x)) - \frac{1}{2} \frac{dQ(g)}{dg} - \frac{1}{4} Q^2(g(x)) \right). \quad (4)$$

Eq. (4) relates the only undetermined function $g(x)$ to the difference of the energy E and the potential $V(x)$. Observing that the energy term E on the left-handside of Eq. (4) represents a constant, the authors of Ref. [7] equated certain terms of the right-handside with a constant to account for it. This results simple differential equations for $g(x)$. The authors in Ref. [7] applied this method to the hypergeometric and confluent hypergeometric function and obtained the solutions of some simple potentials.

Considering the particular example of the confluent hypergeometric function $F(-n, \beta; g(x))$ and introducing the simple $g(x) = \rho h(x)$ substitution we get

$$\begin{aligned} E_n - V(x) &= \frac{h'''(x)}{2h'(x)} - \frac{3}{4} \left(\frac{h''(x)}{h'(x)} \right)^2 \\ &+ \frac{(h'(x))^2}{h(x)} \rho \left(n + \frac{\beta}{2} \right) - (h'(x))^2 \frac{\rho^2}{4} + \frac{(h'(x))^2 \beta}{(h(x))^2} \frac{1}{2} \left(1 - \frac{\beta}{2} \right). \end{aligned} \quad (5)$$

Identifying one of the last three terms on the right-hand side of Eq. (5) with a constant, the three shape-invariant potentials of the confluent hypergeometric case, the three-dimensional harmonic oscillator, the Coulomb problem and the Morse potential, are recovered. These potentials appear in the radial Schrödinger equation, therefore in what follows I shall replace x with r .

3 Generalization of the Harmonic Oscillator Potential

A straightforward way of generalizing the simplest possible solvable potentials to more general ones is identifying combinations of several terms on the right-hand side of (5) with a constant. This procedure recovers the Natanzon confluent potentials [5], the solutions of which contain confluent hypergeometric functions. The most general six-parameter version of these potentials can be obtained by considering the combination of all three terms on the right-hand side of (5), which explicitly contain parameters, however, the technical difficulties increase considerably in this case. The problem remains relatively easy to handle if we take the combination of two such terms only. Considering the differential equation

$$(h'(x))^2 \left(1 + \frac{\theta}{h(x)}\right) = C \quad (6)$$

corresponds to “mixing” the harmonic oscillator and the Coulomb potentials: $\theta \rightarrow 0$ recovers the latter one [6], while $\theta \rightarrow \infty$ combined with $C = \tilde{C}\theta$ yields the former one. (See Eq. (5)). The Coulomb limit has been discussed in detail in Ref. [6], and here we focus on the harmonic oscillator limit. The potential described here and in Ref. [6] is essentially the same for any finite value of θ , nevertheless, it is more convenient to use different notations when we discuss its connection to the two limiting case. In order to make the formalism of the two limits compatible with each other, here we follow the notations of Ref. [6] as closely as possible.

As described in [6], the differential equation (6) can be solved explicitly for the inverse $r(h)$ function only:

$$r = \tilde{C}^{-1/2} \left(\theta^{1/2} \tanh^{-1} \left(\left(\frac{h}{h+\theta} \right)^{1/2} \right) + \left(h \left(1 + \frac{h}{\theta} \right) \right)^{1/2} \right). \quad (7)$$

This function, of course, can be used to determine $h(r)$ as well to any desired accuracy.

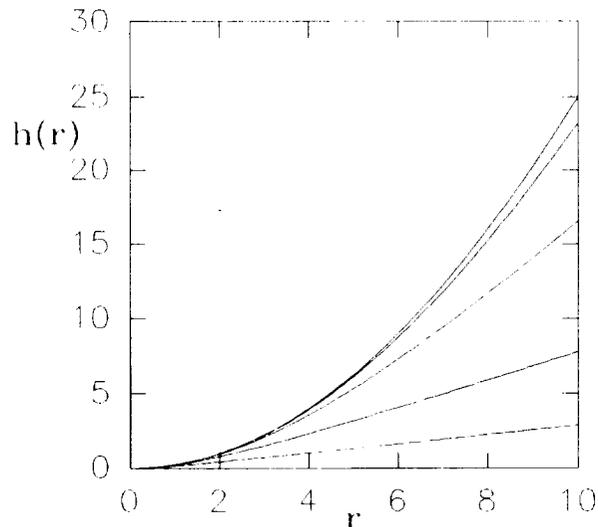


FIG. 1. The $h(r)$ function defined by Eq. (7), displayed for $\theta = 0.1, 1, 10, 100, \infty$ and $\tilde{C} = 1$. (Curves lying higher correspond to higher value of θ .)

We have plotted $h(r)$ in Fig. 1. for several values of the parameter θ . As discussed in Ref. [6], $h(r)$ can be approximated by $\tilde{C}r^2/4$ near the origin, and asymptotically follows $h(r) \rightarrow (\tilde{C}\theta)^{1/2}r$ in the $r \rightarrow \infty$ limit, which correspond to the $h(r)$ functions characterizing the harmonic oscillator and Coulomb problems, respectively. (See e.g. Ref. [8].) The range of the transition between these two regions is governed by the θ parameter: it moves towards larger values of r as θ increases (see Fig. 1.), and disappears completely in the $\theta \rightarrow \infty$ (harmonic oscillator) limit.

Substituting $h(r)$ into (5) and removing the n -dependence from the potential terms by introducing the constant

$$\tilde{D} = \frac{\rho^2}{4} + \frac{\rho}{\theta} \left(n + \frac{\beta}{2} \right) \quad (8)$$

(which amounts to a specific choice of $\rho \equiv \rho_n$.) we arrive at the following potential

$$V(r) = \tilde{C}\tilde{D} \frac{h(r)}{1 + \frac{h(r)}{\theta}} + \left(\beta - \frac{3}{2} \right) \left(\beta - \frac{1}{2} \right) \frac{\tilde{C}}{4h(r) \left(1 + \frac{h(r)}{\theta} \right)} - \frac{3\tilde{C}}{16\theta \left(1 + \frac{h(r)}{\theta} \right)^2} + \frac{5\tilde{C}}{16\theta \left(1 + \frac{h(r)}{\theta} \right)^3} \quad (9)$$

and energy eigenvalues

$$E_n = \tilde{C}(2n + \beta) \left(\left(\frac{1}{\theta^2} \left(n + \frac{\beta}{2} \right)^2 + \tilde{D} \right)^{1/2} - \frac{1}{\theta} \left(n + \frac{\beta}{2} \right) \right). \quad (10)$$

These formulas differ from the corresponding ones in Ref. [6] only in a shift of the energy scale and in the usage of slightly different parameters. The changes reflect the difference between the Coulomb and harmonic oscillator limits of the general problem containing both potentials as a special case. These differences, however, do not essentially influence the form of the wavefunctions:

$$\begin{aligned} \Psi_n(r) = & \frac{\tilde{C}^{1/4} \rho_n^{\frac{\beta+1}{2}}}{\Gamma(\beta)} \left(\frac{\Gamma(n + \beta)}{n!((\beta + 2n)\theta^{-1} + \rho_n)} \right)^{1/2} \\ & \times (1 + h(r)/\theta)^{1/4} (h(r))^{\frac{2\beta-1}{4}} \exp\left(-\frac{\rho_n}{2} h(r)\right) F(-n, \beta; \rho_n h(r)). \end{aligned} \quad (11)$$

(Here and in Eq. (10) n denotes the number of nodes in the radial wavefunction.)

As we can expect from (6), these formulas reduce to the corresponding ones for the harmonic oscillator in the $\theta \rightarrow \infty$ limit, if we introduce the notation $\omega = \tilde{C}\tilde{D}^{1/2}$ and $l = \beta - 3/2$. In particular, the two last terms in (9) vanish and the first and second terms transform into the harmonic oscillator and centrifugal terms, respectively. We have displayed $V(r)$ and the position of some of the lowest-lying energy eigenvalues in Fig. 2. for some values of parameter θ . As it can be seen there, the oscillator character of the potential strenghtens with increasing θ . $V(r)$ is oscillator-like near the origin, and approximates the Coulomb potential (with $Ze^2 = \tilde{C}^{1/2}\tilde{D}\theta^{3/2}$) for large r . The domain of oscillator-like behavior expands with increasing θ : this is related to the structure of $h(r)$ discussed previously. (See also Fig. 1.) Also, the energy spectrum is oscillator-like for small values of n , and Coulomb-like for large n : E_n converges to $E_{n \rightarrow \infty} = V(r \rightarrow \infty) = \tilde{C}\tilde{D}\theta$. See Ref. [6] for a more detailed description of $V(r)$ in terms of powers of r .

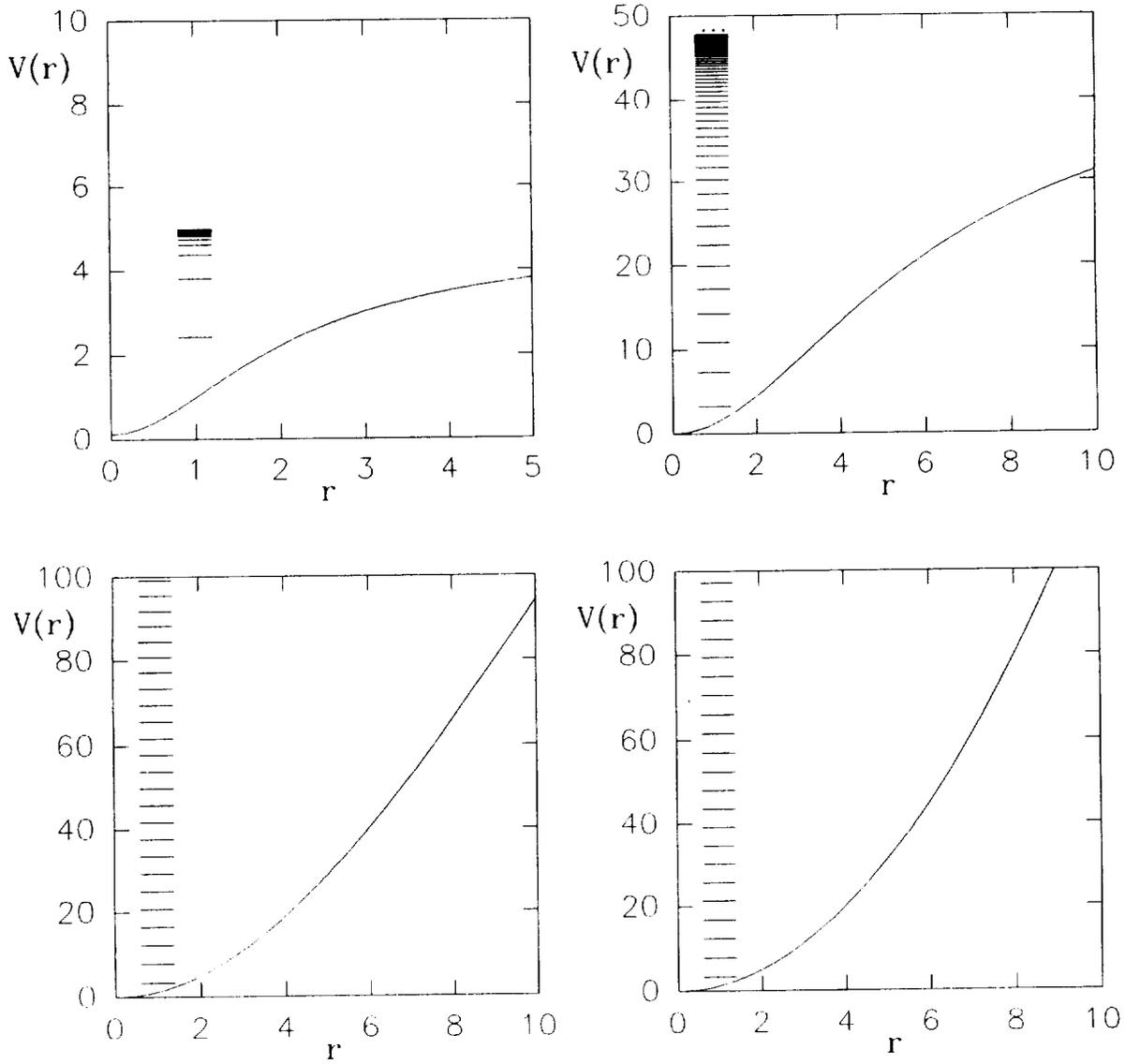


FIG. 2. Potential $V(r)$ in Eq. (9) displayed together with the lowest-lying energy levels for $\theta = 1, 10, 100$ and ∞ . The other parameters are $\tilde{C} = 1$, $\tilde{D} = 5$ and $\beta = 1.5$ in all cases. $V(r \rightarrow \infty) = \tilde{C}\tilde{D}\theta$ in each case.

Similarly to the Coulomb limit discussed in [6], this potential can be rewritten into the sum of a central, centrifugal and l -dependent part:

$$V(r) = V_0(r) + V_l(r) + \frac{l(l+1)}{r^2}, \quad (12)$$

where

$$V_0(r) = \tilde{C}\tilde{D}\frac{h(r)}{1+h(r)/\theta} - \frac{3\tilde{C}}{16\theta(1+h(r)/\theta)^2} + \frac{5\tilde{C}}{16\theta(1+h(r)/\theta)^3}, \quad (13)$$

and

$$V_l(r) = \frac{l(l+1)}{r^2} \left(\frac{\tilde{C}r^2}{4h(r)(1+h(r)/\theta)} - 1 \right) \equiv \frac{l(l+1)}{r^2} v(r). \quad (14)$$

The definition of l is, however, different in the two limits: $l = \beta/2 - 1$ for the Coulomb [6], and $l = \beta - 3/2$ for the oscillator limit. (Fig. 2. displays potentials with $l = 0$ only.) Also in contrast with the Coulomb limit, $v(r)$ in Eq. (14) does not vanish for large values of r , rather it goes to the value $-3/4$. This, again, is the consequence of the asymptotical Coulomb-like character of $V(r)$.

It is remarkable, that E_n depends on the combination $2n + \beta$ only (i.e. on $2n + l + 3/2$ in the oscillator limit), therefore the generalized harmonic oscillator potential has a degeneracy pattern similar to that of the harmonic oscillator. In other words, the terms representing the anharmonicity do not remove the degeneracy of the energy levels.

This generalization of the harmonic oscillator potential could be applied to physical problems, where an attractive Coulomb potential is distorted by an oscillator-like potential component for small values of r . This is the case, for example, for a finite, homogenous, spherical charge distribution, but in that case the resulting potential can strictly be separated into two domains, where it *exactly* follows r^2 -like and r^{-1} -like behaviour. The potential discussed here can be considered a deviation from this simple model problem. An example for a similar situation is discussed in Ref. [9] in connection with a potential experienced by electrons in certain crystal environments.

Finally, there are some other potentials occupying a similar intermediate position between the simple shape-invariant potentials and the general Natanzon potentials. Some of these, like the Woods-Saxon [10] and Ginocchio [11] potentials have been found earlier, while some others, the "PII" [12] potential and those in Refs. [13,14,15] have been identified only recently, mainly in SUSYQM-related studies. See Ref. [6] for more details.

4 Summary

Here I have analyzed a four-parameter potential, which contains both the harmonic oscillator and the Coulomb potential as special cases. I have interpreted this potential as the generalization of the harmonic oscillator potential, and have established that it is a special admixture of a long-range attractive Coulomb term, and an oscillator-like term near the origin. This is also reflected in the structure of the energy spectrum.

Exact analytical solution of the radial Schrödinger equation can be obtained for any partial wave, however, an angular-momentum-dependent term appears for $l \neq 0$. A remarkable finding is that the anharmonicity appearing in the general form of the potential does not remove the degeneracy of the energy levels.

Similar generalizations of the harmonic oscillator and other well-known potentials are also possible by considering further simple differential equations similar to that in Eq. (6). These subclasses of the Natanzon potentials seem to be suitable for applications, because they have

more flexible shape than the simplest solvable potentials, but may still remain relatively simple to handle mathematically.

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